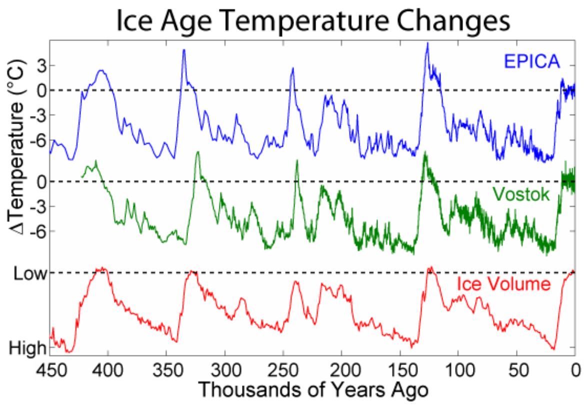
Mathematics and climate change: What role do you think mathematics can play in guiding policy makers and in helping public understanding?

Climate change is the biggest challenge our planet faces today. From glacier retreat to sea level rise to extreme weather events, no corner of our world is safe from its effects. Galileo once wrote that the universe is “written in the language of mathematics” [1]. If this is true, then perhaps mathematics holds the solution for preventing climate change. Over the years, climatological technology has advanced greatly alongside the urgency to assess the severity of our impact on the climate. This essay will provide an introduction to climate change, the fundamentals of climate models, as well as the challenges mathematicians face in bridging the gap between science and policy decisions and in developing public understanding.

An Overview on Climate Change:

Climate change is the long-term, average shift in weather patterns. Today, the primary cause is the emission of greenhouse gases, particularly carbon dioxide, methane and nitrous oxides. These gases are released from burning fossil fuels, cattle ranching, deforestation and so on.

Since the Industrial Revolution began in the 18th century, such activities have skyrocketed and, subsequently, greenhouse emissions have risen to dangerous levels. Due to their specific quantised energies, these gases effectively absorb infrared radiation which creates a heat-trapping effect [2]. This results in the sunlight radiated by Earth back into space to be blocked by the atmosphere, forming a blanket of insulation that raises the temperature of the Earth [3].



Mathematicians can use data and modelling to assess the consequences of global warming. In order to predict future trends, scientists look to the past. For example, ice cores from the Arctic show how carbon dioxide levels have changed over time. Graphs generated from ice core data [4] show a consistent glacial cycle and, remarkably, that the atmosphere should be cooling at present [5]. For millions of years, cyclical variations in the Earth’s atmosphere have occurred due to physical mechanisms such as the Milankovitch Cycles; these are what cause ice ages and inter-glacial periods. It is also worth noting that without the natural greenhouse effect, the temperature of the Earth would be far too low for life to exist. Yet, the extreme changes we see today seem to be outside the bounds of natural variation [6].

The UK Met Office Hadley Centre set the Earth’s climate to that at the start of the twentieth century and ran it through a model to predict trends in the following century, once with stable carbon dioxide levels and again with the actual measured increase. This showed that the Earth ought to be cooler than it is today, proving that this enhanced greenhouse effect is solely due to human activity [4]. Therefore, solutions must come from us as well, and quickly.

In recent years, the momentum for climate activism has reached a peak, and the pressure on politicians to act is mounting. To design international policies that fit the agendas of individual nations and which have a high chance of preventing serious damage to the environment, we turn to mathematicians to design mathematical models on supercomputers that generate accurate predictions.

However, there are significant challenges mathematicians face in this. Firstly, the climate is a chaotic system where there are hundreds of millions of variables that affect each other in non-linear feedback relationships, generating billions of degrees of freedom in possible outcomes. It is a fact that the weather cannot be predicted more than a week in advance. Secondly, modelling every single point in the atmosphere with partial differential equations is impractical, so instead the atmosphere is split into small grid cells and average trends are calculated in each. As Niels Bohr once said, "It is difficult to predict anything, especially about the future", and this is all too true in the case of climate modelling. The uncertainty in predictions is currently impossible to eradicate, and could no doubt be a cause for people to question the reliability of climate models [4].

Building a Climate Model:

In essence, a climate model is a computer simulation of the Earth's climate, running on huge quantities of data to study and explain the effects of different variables and make predictions. Though these models are exceedingly complicated and contain billions of lines of code, they can be stripped down to their essence: the laws of physics. Mathematicians take simple laws of thermodynamics and motion and manipulate them into explaining how our climate works.

A well-known example is the Energy Balance Model (EBM). The EBM is a highly simplified climate model which states that the energy received by the Earth from the Sun is balanced by the energy the Earth radiates back into space. This balance is important as the Earth's systems may not be able to cope with extreme imbalances.

We assume that the Earth is a single body with temperature T. The heat absorbed by the Earth can be represented as $(1-\alpha)S$, where α is albedo, or how reflective the Earth is (0.31) and S is the solar constant, or the average power hitting the Earth's atmosphere (342 Wm^{-2}).

The heat radiated back can be shown as $e\sigma T^4$, where e is emissivity, or how well the Earth emits energy as thermal radiation (0.605) and σ is the Stefan-Boltzmann constant (5.67×10^{-8} $Wm^{-2}K^{-4}$)

We can equate the two to create a state of global heat balance: $e\sigma T^4 = (1-\alpha)S$.

Solving for temperature we get the energy budget equation: $T = \sqrt[4]{\frac{(1-\alpha)S}{e\sigma}}$

Observe how each variable affects temperature. The solar constant changes with the Earth's orbit so we can discount this in order to focus on human impacts. As CO_2 levels increase, emissivity decreases which causes a rise in temperature. Albedo is affected by soil, vegetation etc. The North and South poles have a high albedo due to their snow and ice, which reflect the majority of the Sun's radiation to maintain global heat balance. Temperature increase melts the ice, reducing the Earth's albedo which consequently increases temperature. This is an example of a positive feedback loop [7].

These non-linear relationships between variables are what cause 'tipping points' in the climate, where a tiny shift can push the Earth's systems into abrupt or even irreversible change. A seemingly minuscule rise in global temperature could plunge the Earth into an entirely different climate state, so suddenly that the Earth's systems would not be able to cope with the change [4]. In order to prevent us from reaching a tipping point, we must first fully understand the mathematical relationships between climate variables.

Computational Fluid Dynamics:

One of the fundamental components of climate models is fluid dynamics, which helps predict water currents and forecast the weather. The Navier-Stokes equations describe how temperature, velocity, density and pressure of a moving fluid are related. They assume that the fluid is incompressible, Newtonian (shear rate does not affect viscosity), and isothermal (no change in heat).

The first equation states that mass is conserved:

$$\nabla \cdot \underline{u} = 0$$

Where u is the velocity vector of the fluid. It has three components u, v, w in the x, y and z directions respectively. ∇ (nabla), is the divergence operator. This tells us to differentiate the components of velocity separately, to see how each is changing with respect to its direction.

$$\nabla \cdot \underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Since the partial differentiation of velocity is 0, the change in mass must also be 0.

The second equation states that momentum is conserved (Newton's Second Law):

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \mu \nabla^2 + \rho \underline{F}$$

This equation can be expanded into a complex system of three differential equations.

For these equations we work with density instead of mass (assuming that volume is constant). So, the left term $\rho \frac{D\underline{u}}{Dt}$ is the time derivative of velocity (acceleration) multiplied by density.

Therefore, the terms on the right represent all the forces being applied to the fluid.

$$-\nabla p + \mu \nabla^2 + \rho \underline{F} = F_{total}$$

First we have the gradient of pressure $-\nabla p$, then the viscosity $\mu \nabla^2$, and finally the external forces acting (i.e. gravity) $\rho \underline{F}$.

To use these equations in modelling, many approximations and assumptions have to be made. However, not only is it extremely complicated to solve these without any approximations, but it has also not been proven that smooth solutions for every point in space even exist. There is a prize of a million dollars for anyone who is able to do so [8]. Einstein's notion that "As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality" [9] summarises the problem perfectly.

The Navier-Stokes equations are extensions of the Euler Equations, which also explain the motion of fluids whilst ignoring viscosity, for example Euler's Equation of Motion:

$$\frac{\partial p}{\rho} + gdz + v\partial v = 0$$

The derivation for this equation is as follows:

Consider a cylindrical element in a flow stream. S is direction of flow, ds is the length and dA is the cross-sectional area. pdA is the pressure force in direction of flow and $\left(p + \frac{\partial p}{\partial s} ds\right) dA$ is the pressure force opposite to direction of flow. The weight is described as $\rho g dAds$ where $\rho dAds = \rho V = mass$. Now consider Newton's 2nd Law $F = ma$. The total force in line with the flow is:

$$pdA - \left(p + \frac{\partial p}{\partial s} ds\right) dA - \rho g dAds cos\theta \text{ (1)}$$

This is equal to ma, or $(\rho dAds)a_s$ where a_s is the acceleration.

Acceleration is the time derivative of velocity $\frac{dv}{dt}$, but velocity in turn is the time derivative of displacement $\frac{ds}{dt}$.

So, using the chain rule: $a_s = \frac{\partial v}{\partial s}\left(\frac{ds}{dt}\right) + \frac{\partial v}{\partial t}$

However, $\frac{ds}{dt} = v$ and, assuming steady flow, $\frac{\partial v}{\partial t} = 0$

Therefore, $a_s = \frac{v\partial v}{\partial s}$ (2)

Substituting (2) into (1):

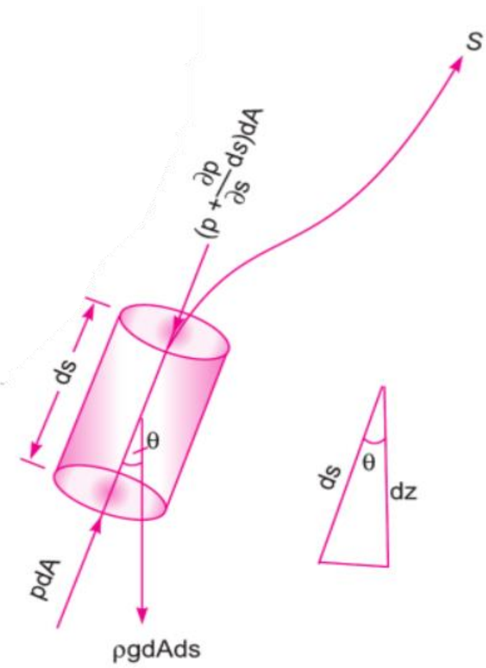
$$-\frac{\partial p}{\partial s} dsdA - \rho g dsdA cos\theta = \rho dsdA \frac{v\partial v}{\partial s}$$

Dividing by $\rho dsdA$: $-\frac{\partial p}{\rho \partial s} - g cos\theta = \frac{v\partial v}{\partial s}$

Rearranging the terms: $\frac{\partial p}{\rho \partial s} + g cos\theta + \frac{v\partial v}{\partial s} = 0$

Using trigonometry, $cos\theta$ can be written as $\frac{dz}{ds}$

So, $\frac{\partial p}{\rho \partial s} + g\frac{dz}{ds} + \frac{v\partial v}{\partial s} = 0$



This simplifies to reach Euler's Equation of Motion [10]:

$$\frac{\partial p}{\rho} + gdz + v\partial v = 0$$

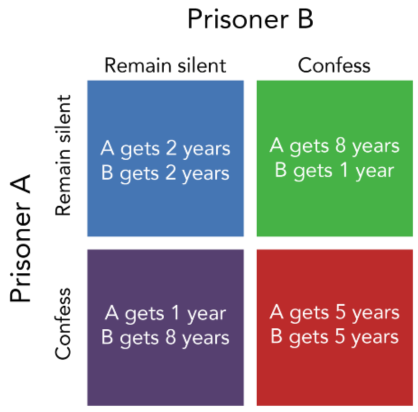
Such equations are the skeleton of a complex climate model. From here, millions of variables are incorporated into the code until an accurate model is reached. The development of climate models over the past 50 years is remarkable. In the 1970s, models only contained a limited number of variables such as carbon dioxide and rain, and the lack of computer power meant that these models could only be run for short periods of time. The Met Office model HADCM3, made in 1999, was the first not to require flux adjustments (artificial adjustments to prevent unrealistic climate predictions) [11]. Today, countries across the globe have developed models such as NCAR from US and CCSR/NIES from Japan that can predict climate trends up to 1000 years ahead with incredible accuracy [4].

Policy and Public Understanding:

Mathematicians generate the knowledge, but politicians and the public have the power to act on it. It is absolutely vital therefore, that we bridge the gap between science and policy making. The international politics of climate change presents a challenge to effective climate change action. Politics may appear to be far from the realms of science and logic, but underlying political decisions is behavioural mathematics that can be analysed and predicted.

Game Theory is a mathematical framework for analysing strategies among competing players and is used to predict outcomes in a variety of circumstances, including politics. A simple example is the Prisoner's Dilemma, which explores the different ways an individual can approach a decision where the outcome is dependent on another.

Two thieves, A and B, have been arrested for robbing a bank. They are kept in separate isolated cells, and each person cares more about their personal freedom than the welfare of the other. The prosecutor gives prisoner A an offer: If they confess and prisoner B remains silent, A will serve one year and B will serve eight years. If they keep silent and prisoner B confesses, B will serve one year and A will serve eight years. If both of them confess, they will each serve five years. If both keep quiet they will each serve two years. The prosecutor then gives prisoner B the same offer.



When reasoning the best possible outcome, a conflict is reached: whether to go for the best overall outcome where both remain silent but are forced to rely on the other player choosing the same, or to pick the outcome that will be best for themselves regardless of what the other player chooses. The second option is called a Nash Equilibrium, which in this case would be for each of them to confess. By choosing their own self-interest they will have avoided the best and worst outcomes [12].

This sort of behavioural science is seen constantly in global politics, where countries have their own agendas and want to avoid being in a dependent position. So how can we combine the moral duty of climate action with the opportunist nature of politics?

Game theorists could play a role in advising politicians on the smartest approach at reaching an international agreement that considers the interest of individuals as well as the overall outcome. Climate change is a global challenge, and the solution must be so too if we want to avoid jeopardising the long-term future of our planet for the sake of short-term self-interest.

Generating the data to reach informed decisions is only half of the battle; raising awareness of it is equally necessary. It is crucial that the impacts of climate change are made tangible to the public. Average global temperatures have increased by over one degree since 1880 and could increase by up to four degrees by the end of the century, according to Met Office [13]. This information alone is practically meaningless to non-scientists and would encourage little change. However, the fact that this temperature increase likely caused the 2003 European heatwave that killed 20000 people [14], would have a far greater impact on public understanding. The scientists and mathematicians of the world are now collaborating to provide climate discussions to an international audience. 'One World Mathematics of Climate' is one such collaboration which holds monthly seminars on the topic. The Cambridge Zero climate change initiative of the University of Cambridge aims to develop solutions for our society and economy by engaging with passionate and intelligent young people. If mathematicians continue to act as spokespeople for climate action then we will have a much better chance at preventing the future that mathematicians have already predicted.

Conclusion:

Mathematicians are absolutely integral in every step towards preventing climate change. From designing climate models and presenting results that can be understood by the public to guiding policy makers into achieving the most effective outcome, the mathematics involved in the climate crisis is endless. There is still much that is unknown about the future of our planet, and there are still those who deny the threat of climate change entirely. In the words of Stephen Hawking, "We are in danger of destroying ourselves by our greed and stupidity. We cannot remain looking inwards at ourselves on a small and increasingly polluted and overcrowded planet" [15].

If we do manage to slow down, or even reverse the damages caused by climate change, our unsung heroes will surely be the mathematicians.

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