Discuss the fundamental mathematical principles underlying gambling/betting, how the theory came up and how it developed through time. Do these principles have applications in other areas?

The vital mathematical principle underlying all gambling and betting is probability theory. In fact, these 'games of chance' are the very reason for this mathematical principle's existence. The incredible importance of probability in our everyday lives factors into every decision we make on a daily basis, as Gerda Reith in 'Age if Chance' (1999) emphasises: "We are all gamblers" and in the modern era "chance has become an irreducible aspect of daily life: risk, speculation, indeterminism and flux are our constant companions in social, economic and personal affairs."

The development of probability and statistics started between the 8th and 13th Centuries with Middle Eastern mathematicians studying cryptography. Al-Khalil wrote the 'Book of Cryptographic messages' which contains the first use of permutation and combination formula to list all possible Arabic words with and without vowels.[5] These combinatoric formulas were officially proven by Jakob Bernoulli in his 1713 book 'Ars Conjectandi'.

Permutations are possible arrangements of r members in a set of n distinct elements, r being an integer with $1 \leq r \leq n$. The formula for total number of r-permutations is:

$$P_{(n,r)} = nPr = \frac{n!}{(n-r)!} = n \times (n-1) \times \ldots \times (n-r+1)$$

Proven by showing that if the first element of the permutation can be chosen in n ways (there are n elements), there are $n-1$ ways to choose the second element (there are $n-1$ elements left) and so on until there are exactly $n-(r-1) = n-r+1$ ways to choose the rth element.

While its direct usage is rare in betting, this formula is used in scenarios such as horse racing – an 'exacta' or 'perfecta' is a wager that wins if the race's first two finishers are picked in exact order; for example with 5 horses with equal chance: $P_{(5,2)} = \frac{5!}{3!} = 60 =$ odds of winning are 1 in 60.[1]

Combinations are selections of r members of a set of n elements regardless of order. Its formula for total possible r-combinations is: $C_{(n,r)} = nCr = \frac{n!}{r!(n-r)!}$

The r-permutations of a set can be obtained by forming the r-combinations and ordering each of the elements in each r-combination, which can be done in $P_{(r,r)}$ ways. Using the product rule for counting: $P_{(n,r)} = C_{(n,r)} \times P_{(r,r)}$

Therefore, $$C_{(n,r)} = \frac{P(n,r)}{P(r,r)} = \frac{n!}{(n-r)!} \div r! = \frac{n!}{(n-r)!} \times \frac{1}{r!} = \frac{n!}{r!(n-r)!}$$

Use of combinations are very important in a variety of gambling games such as lotteries: e.g. the National Lottery draws 6 numbers from 1 to 59. The calculation to find the odds involved in winning would be $\binom{59}{6} = \frac{59!}{53!6!} = 45{,}057{,}474$ combinations = 1 in 45 million odds. However, the combination formula can also be applied to modern casino gambling games e.g. poker, where it can be used to calculate the probability of being dealt a royal flush in 5 cards: number of 5 card hands that can be dealt from 52 deck of cards $= C_{(52,5)} = \frac{52!}{47!} \times 5! = 2{,}598{,}960$ combinations; since 4 of these combinations are royal flushes, you divide the number of combinations by 4 making the odds 1 in 649,740.[1]

The roots of modern probability theory lie in the 16th and 17th century, when gamblers started to turn to mathematics for answers on how to make money. In the 16th century, Italian mathematician Gerolamo Cardano gave the first definition of classical (mathematical) probability in his book ‘De Ludo Aleae’: “we should consider the whole circuit, and the number of those casts which represents in how many ways the favourable result can occur, and compare that number to the rest of the circuit.” The ‘circuit’ is a sample space – the set of all possible outcomes – and Cardano correctly defines the odds of the event by favourable events (r) to unfavourable events (s) corresponding to probability $\frac{r}{r+s}$. He also set out the multiplication rule for independent events ‘Cardano’s formula’, using the example of an event with probability $\frac{1}{3}$ and gives the odds for the event happening thrice in a row as $(3^3 - 1)$ to 1 = 26 to 1.

In 1654, the ‘Two dice problem’ was posed by Chevalier de Méré, a French nobleman and gambler who had made consistent profit betting even money that a 6 would be rolled at least once in four throws of a single die. His incorrect reasoning was that the chance of rolling a six in one throw was $\frac{1}{6} \times 4 = \frac{4}{6}$. He proposed a wager with what he thought was the same probability of winning – that double 6 would be rolled at least once in 24 rolls of a pair of dice ($\frac{1}{36} \times 24 = \frac{24}{36} = \frac{4}{6}$ [1] is the incorrect reasoning in this instance). However, when he continuously lost this second wager, he turned to Pascal and Fermat who solved this dilemma by using the previously stated multiplication rule - for independent events, the probability of all of them occurring equals the product of their individual probabilities; and the complement rule - the probability of an event occurring plus the probability of that event not occurring equals 1. Using these rules, the probability of rolling at least one six in four throws $= P(X \geq 1) = 1 - P(X = 0) = 1 - \left(\frac{5}{6}\right)^4 = 1 - 0.482 = 0.518$

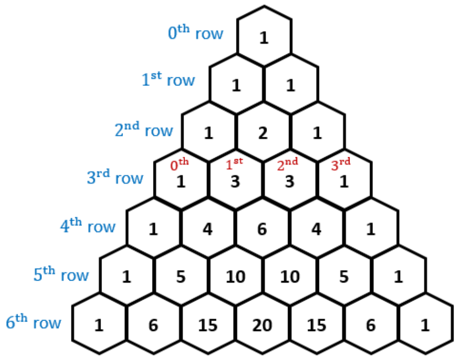
Similarly, the probability of rolling at least one double six in twenty-four throws of a pair of dice $=$ $1 - \left(\frac{35}{36}\right)^{24} = 1 - 0.509 = 0.491$.[1]

The Problem of the Points was posed in the 1494 by a monk called Luca Paccioli, who, drawing on a gambling game called Balla, wondered how the stakes of a fair game should be divided if the game is ended early.[1] A modern version of this question is “A and B are playing a fair game (each person has an equal chance of a point) which continues until one has won 5 rounds. The game actually stops when A has won 4 and B has won 3. How should the stakes be divided?”[2] While the initial reaction to this question was met with claims that it depends on the ethics of the men involved, Pascal and Fermat developed a universally accepted mathematically based answer using the concept of the expectation of each player winning. If player A needs a more rounds to win and player B needs b, the game will surely have to be over after $a + b - 1$ rounds; if the players do play $a + b - 1$ rounds the number of possible outcomes is 2^{a+b-1}. While the game may have been decided earlier than that maximum number of rounds, they play on in this scenario for the sake of clarity. In a fair game the probability of all these outcomes are equal, so Fermat was able to calculate the odds by writing out a table of all the outcomes:

Point 8	Point 9	Overall winner
A	A	A
A	B	A
B	A	A
B	B	B

The result is that A wins $\frac{3}{4}$ times, therefore has a $\frac{3}{4}$ chance of winning and B has a $\frac{1}{4}$ chance of winning therefore the stakes should be split in a 3:1 ratio.

Blaise Pascal looked for a way of generalising this that would avoid tedious listing of probabilities and instead related the problem of points to the arithmetic triangle (the concept of which he did not invent even though we today call Pascal's triangle). Consider, as above, two players A and B, who are a and b points away from winning respectively, he used the row $a + b - 1$ and then counting from the left, the number of ways that A can win 0,1,2… rounds is the number on the corresponding number of the triangle. So player A wins if she wins any of rounds above the number a and Pascal shows that this number is given by the sum of the first b entries. Similarly, player B wins if she wins any of rounds above the number b and Pascal shows that this number is given by the sum of the last a entries. In this way the general division rule stated as odds is[6]



(sum of the first b entries for row $a + b - 1$) to (sum of the last a entries for row $a + b - 1$)

So in reply to the question set above, the odds/ stake division is (1+2=)3 to 1. Using modern notation and the combination operator, his formula for finding the ratio of this without having to physically look at his triangle and merely calculate the numbers on it is

$$\sum_{k=0}^{b-1} \binom{a+b-1}{k} \ to \ \sum_{k=b}^{a+b-1} \binom{a+b-1}{k}$$

The use of binomial coefficients as above in pascals triangle was officially developed into a distribution in terms of probability by swiss mathematician Jakob Bernoulli in 1713 - a formal proof was published posthumously by determining that the probability

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a + b \\
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a+b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5
\end{aligned}$$

of k such outcomes in n repetitions is equal to the kth term (starting with the probability of k is 0) in the expansion of the binomial expression $(a + b)^n$, where a is the probabilities of that outcome and $b = 1 - a$; (the fact of it only having two possible outcomes gives it the name binomial distribution). Using the example of a die, the probability of getting any number on each roll is $\frac{1}{6}$. The probability of turning up 2 sixes in 5 rolls, then, is equal to the 2nd term (starting after the 0th term) in the expansion of $(\frac{5}{6} + \frac{1}{6})^5 = 10a^3b^2 = 10(5/6)^3(1/6)^2 = 0.16075$. In modern notation using the combination formula this is as follows: $P(X = k) = \binom{n}{k} a^k b^{n-k}$

In terms of casino games, this can be used to find the probability of winning at least 1 out of 3 games when you bet on red on a double-zero roulette, the probability of winning is $\frac{18}{38}$ and losing $\frac{20}{38}$ and taking into account that $P(X \geq 1) = 1 - P(X = 0)$, this gives us the sum:

$$1 - \binom{5}{0} \left(\frac{18}{38}\right)^0 \left(\frac{20}{38}\right)^3 = 1 - \left(\frac{20}{38}\right)^3 = 0.8542$$

We can use the double-zero roulette to examine a few other important mathematical factors involved with gambling such as expected value: 'The average amount a player can expect to win or

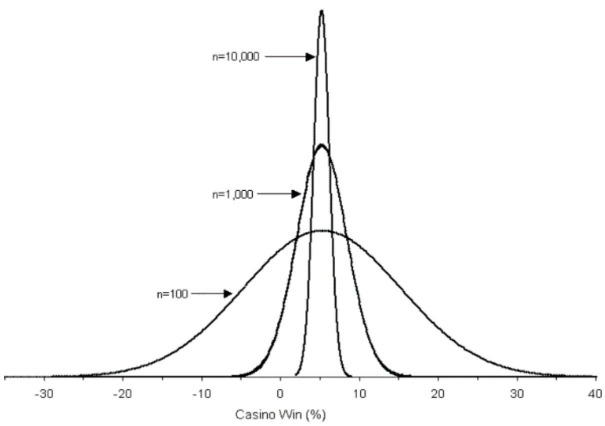
lose in the long run', calculated by the sum of all possible probabilities multiplied by their associated gains or losses. The first occurrence of this formula and its proof was by Christian Huygens, in the third proposition of his treatise, stating "my expectation will be worth ($\frac{ap + bq}{p+ q}$)".[3]
In n terms: $\frac{a_1p_1+a_2p_2+\cdots+a_np_n}{p_1+p_2+\cdots+p_n}$, where a_x's are the winnings and p_x's are the odds. In roulette, if you bet £100 on a single number: $\left(\frac{1}{38} \times £3500\right) + \left(\frac{37}{38} \times -£100\right) = -£5.26$. The expected value of -£5.26 means that in the long run you will lose £5.26 for every £100.[1]

The negative percentage form of the expected value is the house advantage, e.g. here it would be 5.26%. This illustrates to the casino the value of that particular game by showing how much they can expect to retain in the long run proportionally.[1] House advantage can be calculated directly from payoff and true odds. Payoff odds: x to z - True odds: y to w - House advantage = $\frac{yz - xw}{z\,(y + w)}$

True odds means the probability of a win occurring whilst payoff odds are actually what you receive on top of your wager. For a roulette bet on a single number: True odds = 37 to 1; Payoff odds = 35 to 1; House advantage $= \frac{37 - 35}{1\,(37 + 1)} = 0.0526$ or 5.26%.

The mean of the amount won in roulette approaches the expected value the more rounds that are played by the law of large numbers. However, Central Limit Theorem developed by Abraham de Moivre in 1733 is what allows us to quantify the speed of convergence, telling us how the risk of return differs the more rounds we play (as shown on the right). CLT states that the mean of a large sample taken from any random variable is always approximately normally distributed – if $X_1, X_2, \ldots, X_n$ is the random sample of size n from a population with mean μ and variance σ^2, then $\overline{X}$ is approximately $\sim N(\mu, \frac{\sigma^2}{n})$; due to the law of large numbers, as n gets larger, this approximation gets better.

FIGURE 1



For example, a £5 even money bet on red on the roulette from the casinos perspective; expected value $= \left(5 \times \frac{20}{38}\right) + \left(-5 \times \frac{18}{38}\right) = £0.263$:

First we calculate variance and standard deviation (square root of variance) of a single wager. The variance is stated as $Var(X) = \sigma^2 \sum[(x_i - \mu)^2 \times P(X = x_i)]$. In this case:
$Var(X) = (\frac{20}{38})(5 - 0.263)^2 + (\frac{18}{38})(-5 - 0.263)^2 = £24.93074795$ and $\sigma = £4.993070$.

Due to the central limit theorem, we can use the 'empirical rules' of the normal distribution to apply confidence limits to many repetitions of this bet or wager. For example for 1,000 repetitions:

$Expected\ win\ (EV) = £263.16\ (£0.263 \times 1000)$ $\sigma\ win = £157.89\ (£4.993070 \times \sqrt{1000})$

As 68% of the values are within 1 σ of the EV, in 1000 repetitions of this bet 68% of the time the casino will win within £421.05 and $105.27. 95% are within 2 standard deviations so 95% of the time the casino will win between $578.94 and $-52.62.

Away from gambling, probability principles are a vital instrument when investing in the stock market. The potential returns for a stock are modelled by the normal distribution, meaning it is easy to use the empirical rules of the normal distribution to compare risk and return – best case is low risk and high return. For example, if the mean daily % price change is 1.5% and the standard deviation is 1%, 68% of the time you can expect potential returns of the stock to be between 0.5% and 2.5%; 95% of the time you can expect potential returns to be between -0.5% and 3.5%. This is a much riskier investment than say one with a standard deviation of 0.1%.[9] Financial analysts will even utilise the central limit theorem, for example, when an investor would like to estimate the overall return for a stock index that comprises of 1,000 equities.[10]

When building a portfolio of investments, professional fund managers use a mathematical approach called modern portfolio theory (MPT) which is also founded on the concept of normal distribution – it aims to maximise expected return, for a given amount of portfolio risk by selecting the proportions of various assets while keeping the investments diverse. It uses the fact that R_p (expected return of portfolio) is normally distributed: $R_p \sim N(\mu_p, {\sigma^2}_p)$

The formula for R_p is similar to gambling EV where w_i is the proportionate weight of asset i, R_i is the expected return of asset i: $R_p = \sum w_i R_i$

Standard deviation or 'portfolio risk' is then calculated, using the formula:

$$\sigma_p = \sqrt{\sum_i \sum_j [w_i w_j \, \sigma_i \sigma_j (cor - cof_{ij})]}$$

where $(cor - cof_{ij})$ is the correlation co-efficient between expected returns of assets i and j. I.e. using negatively correlated assets increases diversity and lowers σ_p.

For example, if you were to decide how much capital should be allocated to two available assets (a and b below) so that the expected return is maximised and risk is lowered:

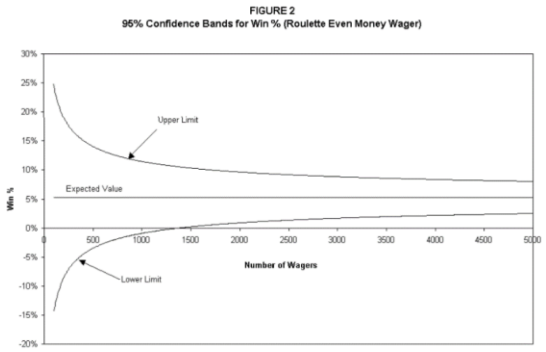
$R_a = 0.175$ $R_b = 0.055$ $\sigma_a = 0.258$ $\sigma_b = 0.115$

$\sigma_{ab} = -0.004875$ $(cor - cof)_{ab} = -0.164$

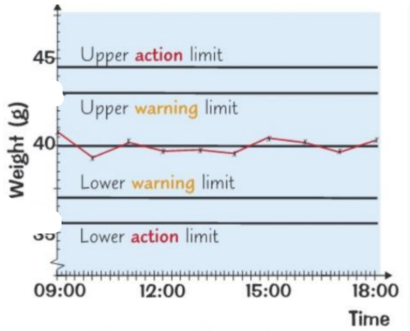
When allocation is even, the $R_p = 0.115$ and $\sigma_p = 0.1323$; showing us that for this allocation, return as well as risk is midway between individual values of each asset.

If a 1.5 capital allocation to asset a and a -0.5 capital allocation to asset b (through shorting the stock) was used, $R_p = 0.1604$ and $\sigma_p = 0.4005$, a much higher expected return but at a much higher risk.[9]

The use of confidence intervals as 'control limits' on a control chart is a statistical process control tool for quality assurance on production line or equivalent. On a production line, the most common 'limits' are warning limits at 2 standard deviations (the equivalent of a 95% confidence band), within this the process is working as it should, and action limits at 3 standard deviations – if a recording is outside of this, the manufacturing line is stopped immediately.[12] These variations are measured using control charts that have similarities to the graphs that some casinos use to make sure they are not being cheated as shown below[1]



Production Line
Control chart



To conclude, the fundamental mathematical principle of probability underlying gambling is multifaceted and its many theories have been developed through thousands of years – although gambling itself gave way to probability's breakthrough in the 17th century during the correspondence of Pascal and Fermat which lead to the first real 'probability' text by Bernoulli which integrated his own theories with theirs. While these principles at first arose though use in games of chance, in our modern, complex era we are finding more and more applications for probability principles, such as financial stock analysation and quality control on production lines.

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